

Student seminar exercise sheet Week 9

1. Show that J_F is a topological group when equipped with the restricted product topology. Show furthermore that it is a locally compact group.
2. Show that F^\times is a discrete subgroup of J_F . (Recall that we view F^\times as a subgroup of J_F via $\iota(F^\times)$.)
3. In this exercise, you will prove the finiteness of the ideal class group \mathcal{C}_F . Before we do so, we introduce the content map as follows:
For an idèle $\mathbf{a} = (\dots, a_v, \dots) \in J_F$, define its content to be

$$\text{content}(\mathbf{a}) = \prod_{v \in V_F} \|a_v\|_v$$

You may assume that the content map is a continuous homomorphism $J_F \rightarrow \mathbb{R}_+^\times$.

- (a) Denote the kernel of the content map on J_F by J_F^1 . Show that $F^\times < J_F^1$ and that the quotient J_F^1/F^\times with the quotient topology is compact.
 - (b) Let $\eta : J_F \rightarrow \mathcal{I}_F$ be the map $\mathbf{a} = (\dots, a_v, \dots) \mapsto \prod_{v \text{ finite}} \mathfrak{p}_v^{\text{ord}_v(a_v)}$. Show that if \mathcal{I}_F is given the discrete topology, η is continuous.
 - (c) Show that $\eta(J_F^1) = \mathcal{I}_F$.
 - (d) Show that $\mathcal{C}_F = \mathcal{I}_F/\mathcal{P}_F$ is compact. Finish the argument by remembering that it is a discrete group.
4. This exercise is to get familiar with idèles, and have some glimpses of concrete computations.
Define the map:

$$\begin{aligned} \mathbb{Q}^\times \times \mathbb{R}_+ \times \prod_p \mathbb{Z}_p^\times &\rightarrow J_{\mathbb{Q}} \\ (r, t, (u_p)_p) &\mapsto (rt, ru_2, ru_3, ru_5, \dots) \end{aligned}$$

- (a) Prove that the map is surjective.
Hint: Given $\mathbf{a} \in J_{\mathbb{Q}}$, observe the element $a = \text{sign}(a_{\infty}) \prod_p p^{\text{ord}_p a_p}$.
- (b) Prove that the map is injective.
Hint: It is enough to show uniqueness of the factorization of $1 = (\dots, 1, \dots)$.
- (c) Show that the group on the left is a topological group by exhibiting a fundamental system of neighborhoods of 1.